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# Optimal Pooling of Inventories with Substitution: a literature review

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**Abstract** In many inventory management systems, some kind of substitution flexibility exists, meaning that a substitute or more flexible item can be used (at an additional cost) when the preferred product stocks out. Through the use of substitution flexibility, we can take advantage of the risk pooling effect on the flexible item. Since risk pooling reduces total inventory holding costs, a trade-off between inventory holding costs and flexibility costs will determine the optimal inventory control parameters for the different items. In this research paper, we focus on different types of inventory management systems with substitution flexibility, and discuss three methods suggested in the literature (i.e., newsvendor models, simulation and continuous-time Markov chains) in order to optimally exploit substitution flexibility.

## I. INTRODUCTION

Inventory is not only a cost for a company. It is also a way to satisfy customer demand rapidly. Successful inventory management deals with balancing the cost of inventory with the benefits of inventory. One way of reducing inventory costs is by using flexible inventories. Flexibility can be added in a variety of ways, a.o. through the use of common components (e.g. Hillier 1999, Hale et al. 2001), lateral transshipments (e.g. Robinson 1990, Herer et al. 2006) and postponement (Tibben-Lembke and Bassok 2005).

It is well known that the use of a flexible inventory reduces the need for safety stock (provided that demands are not perfectly positively correlated) and, correspondingly, reduces inventory holding cost. The reason is that the demand of multiple end products is pooled on the flexible inventory. This is the concept of risk-pooling.

Nevertheless, risk-pooling usually comes at a cost. The company has to pay for the added flexibility; this “flexibility cost” can boil down to a *product cost premium* (when the flexible product is inherently more expensive to manufacture or purchase) and/or an additional *adjustment cost* (when the product needs to undergo additional processing steps in order to make it “fit for use” when demand arises).

This observation has spurred research on so-called *substitution* systems, in which flexible (and hence, more expensive) stock is only used when the regular (cheaper) item stocks out. This is also referred to as “tailored flexibility” (Chopra and Meindl 2007) or “tail-pooling” (Van Mieghem 2008).

For instance, a multiproduct company can decide to use a (usually higher quality) item as a substitute if inventory of the regular product is not sufficient to satisfy demand (Khouja et al. 1996, Bassok et al. 1999, Liu and Lee 2007). This is referred to as manufacturer-driven one-way substitution (Chopra and Meindl 2007). Table 1 gives some other examples of settings in which substitution is used.

| | **Product cost of substitute > Product cost of substituted product** | **Product cost of substitute ≤ Product cost of substituted product** |
|---|---|---|
| **Adjustment cost =0** | • Manufacturer-driven one-way substitution (Bassok et al. 1999)<br>o Integrated circuits<br>o Steel beams | • Substitution of blood types at a blood bank (Jennings 1973) |
| **Adjustment cost > 0** | • Sheeting paper from different parent rolls, having the same paper grade but different size (Chauhan et al. 2008) | • Tailored postponement for dyeing fabrics (Kambil 2008)<br>• Lateral transshipments (Robinson 1990, Herer et al. 2006) |

Table 1: Examples of substitution

The challenge then is to manage the inventory of regular and flexible items in such a way that the trade-off between expected savings in inventory holding cost and expected increase in flexibility cost is optimally exploited.
As evident from the table, settings may differ in terms of the product cost premium or the adjustment cost: these do not always have to be strictly positive. One can even find examples with no product cost premium and no adjustment cost for the flexible item. Consider a blood bank (Jennings 1973): the product cost is essentially the same for every blood type, and some blood types can be used as a substitute for other blood types at no cost (a person with blood type A (B) can be treated with blood type A (B) and O; a patient with blood type AB can receive blood from every blood type). This setting is atypical: in absence of any flexibility cost, it is trivial from a cost viewpoint that the optimal solution should be to pool all demand on the most flexible item (blood type O), as this allows to fully exploit the risk pooling effect without incurring a cost penalty. However, as a blood bank is dependent on its donors and cannot freely control the replenishment of its stocks, stocking up on other (less flexible) blood types is necessary to ensure customer service.

In general, determining the optimal inventory control parameters in systems with substitution is complex: demands are only "partially pooled" on the inventory of the flexible item, and the amount of demand that can be "rerouted" to the flexible item depends on the order policies of both the dedicated product and the substitute. The optimal inventory control parameters are influenced by many different factors, such as the control policy used (which can be continuous or periodic), the replenishment lead time (deterministic -- zero or strictly positive-- or stochastic), and the demand structure (demand distributions and correlation between the demands).

This paper aims to give an overview of the methodologies that have been used in the literature to optimize inventory control systems with substitution flexibility. The methods used can be divided according to the control policy used, i.e. periodic review versus continuous review. Section II zooms in on the methodologies put forward for periodic review systems, while section III discusses the continuous review setting. Section IV summarizes the main conclusions.

# II. PERIODIC REVIEW INVENTORY SYSTEMS

In a periodic review policy, the inventory position is checked at regular time intervals (referred to as the review period or the review interval), and an order is placed such that the inventory position is raised to the *order-up-to level* (Chopra and Meindl 2007). The order is only received at the end of the replenishment lead time (which may be zero or strictly positive), and demand that cannot be fulfilled is either backlogged (backlog assumption) or lost (lost sales assumption).
The main inventory control parameter that needs to be decided in a periodic review system is the order-up-to level. In the literature, two main methodologies have surfaced: the newsvendor approach (discussed in section II.A), and discrete-event simulation (section II.B). Each section will discuss the relevant papers that use this methodology, and the assumptions of the periodic review setting studied.

## A. Newsvendor models

The basic newsvendor problem (also known as newsboy problem or single period problem) has been described in multiple textbooks (e.g. Zipkin 2000, Heyman and Sobel 1990 end Silver et al. 1998) and papers (e.g. Khouja 1999, Hillier 1999, Van Mieghem 1998). Although this model only considers one period (and therefore, it is not a periodic review inventory system), the basic newsvendor model (section II.A.1) is used as an introduction to the multiperiod newsvendor model (section II.A.2) and the multiproduct model with flexibility (section II.A.3).

### 1. Basic newsvendor model: single product, single period

The basic newsvendor model only considers one single product with a single sales period. Prior to observing realized demand, the decision maker needs to decide on how many items to take in stock. At this moment, only the demand density function *f(z)* and the demand cumulative distribution function *F(z)* are known; the exact demand $z$ that will arise during the sales period is unknown. The order is received by the start of the sales period, and brings the on-hand inventory to a level y. Due to the uncertainty in demand, the decision maker will incur mismatch costs: demand that cannot be fulfilled is lost, and entails an underage cost $c_u$ per unit. Excess inventory at the end of the sales period incurs an overage cost $c_o$ per unit.
The optimal policy for the single item, single period newsvendor is a base stock policy (Zipkin 2000). In a base stock policy, we order the difference between an order-up-to level and the initial inventory (if there is any). If the initial inventory is higher than the order-up-to level, no order is placed.
For the single period case, it is common to assume that there is no initial inventory. The aim is to determine $y$ in order to minimize the expected total mismatch cost over the period, $E[TC(y)]$:

$$E[TC(y)] = c_o E[y-z]^+ + c_u E[z-y]^+ \tag{1}$$

The first term in expression (1) is the total expected overage cost which consists of the unit overage cost multiplied with the expected on hand inventory at the end of the sales period. The second term is the total expected underage cost which can be decomposed in unit underage cost multiplied with the expected unmet demand at the end of the sales period.
Since $E[TC(y)]$is convex in y (Zipkin 2000), it is sufficient to determine the value of y that solves $\partial E[TC(y)]/\partial y = 0$:

$$\partial E[TC(y)]/\partial y = c_o F(y) - c_u(1 - F(y)) = 0$$
$$F(y*) = c_u/(c_o + c_u) \qquad (2)$$

The ratio $\frac{c_u}{c_u + c_o}$ in expression (2) is referred to as the critical fractile. The optimal $y^*$ is then determined as $y *= F^{-1}\left(\frac{c_u}{c_u + c_o}\right)$.

The single period model thus allows us to find a closed-form expression for the critical fractile, depending only on $c_u$ and $c_o$. The kind of costs contained in $c_u$ and $c_o$ depends on the characteristics of the problem setting. In the single-period case, $c_u$ traditionally encompasses the missed margin per unit, while $c_o$ encompasses the unit purchase price minus the unit salvage value.
In the next section, we extend the single-period setting to a multiperiod case for a model with backlogged demand and a model with lost sales.

## 2. Multiperiod newsvendor model with a single product

In a general multiperiod setting, the decision maker decides at the start of every period (i.e., before observing true demand) the number of items he wants to buy at a unit purchasing cost *c(t)* (the replenishment lead time remains zero, so the units are immediately received). Items that remain unsold by the end of the period are carried over to the next period, and incur a unit holding cost *h(t)*. Demand that cannot be fulfilled incurs a unit shortage cost *b(t)* and is either backlogged or lost.
Note that, in a multiperiod setting, the cost parameters (*c(t), h(t)* and *b(t)*) can change over time. Also, the demand distribution may change from period to period.
A multiperiod model is in general much more complex than a single-period model, for the following reasons: (i) every period starts with an inventory *x(t)* which contains leftover stock form the previous periods; (ii) at the start of every period, we have to make a decision of the order up to level *y(t)*, which may differ from period to period; (iii) the quantity ordered in a period influences the quantities that need to be ordered in the next periods, and (iv) it is common to take the opportunity cost of capital into account (by means of a discount factor $\beta$, $0<\beta\leq 1$ (Zipkin 2000)), since costs incurred in the near future weigh more heavily than costs incurred in the more distant future.

As explained below, the exact solution to any multiperiod *finite* time horizon model can be computed by dynamic programming. It can be shown however that the newsvendor solution, which “myopically” considers only the next upcoming period, provides an exact solution for both the *finite* and *infinite* time horizon models when the cost parameters and demand distribution do not change over time.

The multiperiod newsvendor models with backlogged demand and lost sales are explained in section a and b, respectively.

***a. Multiperiod model with backlogged demand***

The multiperiod model with finite time horizon can be formulated as a dynamic programming model (Zipkin 2000).
Consider *E[DC(t,x(t))]* as the expected total discounted cost at time *t* of periods *t, t+1,…, T* (with *T* the last period) given that the inventory at the start of period *t* is equal to *x(t).*

$$E\big[DC\big(t,x(t)\big)\big] = c(t)\big(y(t)-x(t)\big) + h(t)E[y(t)-z(t)]^{+} + b(t)E[z(t)-y(t)]^{+} + \beta E\big[DC\big(t+1,y(t)-z(t)\big)\big] \qquad (3)$$

Further assume that in the last period *T* there is no demand and it is possible to sell the leftover inventory at the purchasing price of that period or to buy inventory to meet the unmet demand:

$$E\big[DC\big(T,x(T)\big)\big] = -c(T)x(T)$$

By using backward induction, it is possible to solve this dynamic programming formulation recursively. We first start with finding the optimal order up to level *y*(T-1)* that minimizes *E[DC(T-1,x(T-1))].* Since at time *T-1*, there is only one period left, we can solve this problem as a single-period case. At time *T-2,* given the optimal solution for *E[DC(T-1,x(T-1))],* we can find *y*(T-2)* that minimizes *E[DC(T-2,x(T-2))]* as a single period case. By going backward until time 0, we find *y*(0)* which solves *E[DC(0,x(0))]* optimally.

An approximate solution can be found by transforming the model in a myopic policy model. A myopic policy only considers the cost of the next upcoming period, and ignores the costs of any future periods. Let *E[MC(t,x(t))]* denote the total cost for period t in the myopic policy model, given a starting inventory *x(t).*

$$E\big[MC\big(t,x(t)\big)\big] = c(t)\big(y(t)-x(t)\big) - \beta c(t+1)E[y(t)-z(t)] + h(t)E[y(t)-z(t)]^{+} + b(t)E[z(t)-y(t)]^{+} \qquad (4)$$

Note that the expected inventory position at the end of the period ($E[y(t)-z(t)]$) is "eliminated" at a discounted unit cost of *βc(t+1).* If on-hand inventory remains at the end of the period, it is sold at a discounted unit salvage value of *βc(t+1).* If the inventory position at the end of the period is negative, the remaining unmet demand is fulfilled (by purchasing units at the end of the period at a discounted unit purchase cost of *βc(t+1)*).
The myopic policy model is convex in *y(t),* and has a closed-form solution:

$$\partial E\big[MC\big(t,x(t)\big)\big]/\partial y(t) = c(t) - \beta c(t+1) + h(t)F(y) - b(t)\big(1-F(y)\big) = 0$$
$$F(y(t)^{m}) = (b(t) - c(t) + \beta c(t+1))/(b(t)+h(t)) \qquad (5)$$

Note that expression (5) determines a (time-dependent) critical fractile of the type

$\frac{c_u(t)}{c_u(t)+c_o(t)}$, as in the standard newsboy model (expression (2)). In this case, the underage cost $c_u(t)$ is given by $c_u(t) = b(t) - c(t) + \beta c(t+1)$ while the overage cost $c_o(t)$ is given by $c_o(t) = c(t) - \beta c(t+1) + h(t)$.

We can see from expression (5) that the order up to level $y(t)^m$ is independent of the starting inventory position $x(t)$. While expression (5) is easy to compute, the order up to level computed with this myopic policy ($y(t)^m$) is only optimal when the cost parameters ($c(t)$, $h(t)$ and $b(t)$) and the demand distributions do not change over time (ie, they remain the same in all future periods) (Heyman and Sobel 1990). Note that, in that case, the order-up-to level is independent of the time period: $y(t)^m = y^m$, for all $t$. When cost or demand distributions do change, the dynamic programming solution might anticipate this change by altering the order up to level in an earlier period. The myopic policy, however, cannot anticipate –hence the name “myopic”.

The myopic policy solution remains exact for the infinite time horizon case, provided that the cost parameters and demand distributions remain the same over time. Note that, in the infinite horizon case, we need to assume strict discounting (i.e., $\beta<1$) (Heyman and Sobel 1990) to prevent that the expected total discounted cost (expression 3) grows to infinity.

### *b. Multiperiod model with lost sales*

In a setting with lost sales, only the last term of the dynamic programming model formulation (expression (3)) needs to be changed (Zipkin 2000):

$$E\big[DC\big(t,x(t)\big)\big] = c(t)\big(y(t)-x(t)\big) + h(t)E[y(t)-z(t)]^+ + b(t)E[z(t)-y(t)]^+ + \beta E[DC(t+1,[y(t)-z(t)]^+)] \quad (6)$$

Unmet demand is lost, so the inventory position cannot become negative. The dynamic programming model can be solved as described in the previous section.

We can again derive a myopic policy solution by only considering the cost of the next upcoming period, ignoring any future periods. Let $E[MC(t,x(t))]$ denote the total expected cost for period $t$ in the myopic policy model, given a starting inventory $x(t)$. For the lost sales case, we obtain:

$$E\big[MC\big(t,x(t)\big)\big] = c(t)\big(y(t)-x(t)\big) - \beta c(t+1)E[y(t)-z(t)]^+ + h(t)E[y(t)-z(t)]^+ + b(t)E[z(t)-y(t)]^+ \quad (7)$$

Note that the inventory position at the end of the period can now only be positive. The optimal order-up-to level can then be obtained as follows:

$$\partial E\big[MC\big(t,x(t)\big)\big]/\partial y(t) = c(t) - \beta c(t+1)F(y) + h(t)F(y) - b(t)\big(1-F(y)\big) = 0$$

$$F(y(t)^m) = \big(b(t)-c(t)\big)/(b(t)+h(t)-\beta c(t+1)) \quad (8)$$

Note that expression (8) again determines a (time-dependent) critical fractile of the type $\frac{c_u(t)}{c_u(t)+c_o(t)}$: in the lost sales case, the underage cost $c_u(t)$ is given by $c_u(t) = b(t) - c(t)$ while the overage cost $c_o(t)$ is given by $c_o(t) = c(t) + h(t) - \beta c(t+1)$. Hence, the lost sales model has a lower underage cost, resulting in a lower critical fractile and a lower order up to level $y(t)^m$ than the backlogged case. It can be shown that the myopic policy is optimal for both the finite and infinite horizon settings, provided that the cost parameters (i.e., $c(t)$, $h(t)$ and $b(t)$) and demand distributions do not change over time. The infinite horizon case requires strict discounting (i.e., $\beta<1$) to ensure that the expected total discounted cost (expression 6) remains finite.

## 3. Newsvendor models with multiple product types and flexibility

This section will cover the literature on newsvendor models that include multiple product types, and flexibility. Table 2 gives an overview of the key references.

| Reference | Kind of flexibility | Time Horizon | Description | Objective |
|---|---|---|---|---|
| Hillier 1999 | Product commonality | Single & multi period | - No product commonality vs. pure product commonality<br>- Discounted costs<br>- Backorder | Minimize sum of purchasing, holding and shortage costs |
| Hillier 2002 | Product commonality | Multi period | - Product commonality as backup<br>- Discounted costs<br>- Lost sales | Minimize sum of purchasing, holding and shortage costs |
| Hillier 2000 | Component commonality | Multi period | - No component commonality vs. pure component commonality<br>- Discounted costs<br>- Backorder | Minimize sum of purchasing, holding and shortage costs |
| Hale et al. 2001 | Component commonality | Single period | - Component substitution | Maximize sum of sales revenue and salvage value minus purchasing costs |
| Van Mieghem 1998 | Flexible resource | Single period | - Determine capacity level of resources | Maximize operating profit minus capacity cost |
| Van Mieghem and Rudi 2002 | Flexible resource | Single period | - Determine capacity level of resources and optimal order policy of raw material<br>- Discounted costs | Maximize revenue minus the sum of shortage cost, holding costs, purchasing costs and investment costs |

Table 2: Newsvendor models including flexibility

As evident from the table, most papers have considered settings in which flexibility is achieved through commonality (either product or component commonality). Only two of these papers (Hillier 2002 and Hale et al. 2001) consider tailored pooling; the others are limited to the no pooling versus full pooling scenarios.
Additionally, two papers with resource flexibility are included in the overview (Van Mieghem 1998, Van Mieghem and Rudi 2002). Although these papers consider flexible resources instead of flexible inventory, parallels can be drawn between the two settings. Indeed, in both situations, the decision on how much to invest (in inventory viz. in capacity) needs to be made before true demand is known. After demand is realized, the allocation decision needs to be made (demand needs to be

allocated to the different types of inventory viz. to the different types of capacities).
All references assume that replenishment lead times are zero. In the multiperiod settings (Hillier 2000, 2002), cost parameters and demand distributions are assumed to remain the same over time. In what follows, we first discuss the references on commonality (section II.A.3.a); next, we cover the references on flexible resources (section II.A.3.b).
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***a. Product or Component commonality***

The impact of commonality on total cost has been examined in multiple papers. A distinction needs to be made between product commonality and component commonality. *Product commonality* introduces flexibility at the level of the end items. With a no product commonality strategy (Figure 1), the demand for every end item is fulfilled by the unique (dedicated) product. With a pure product commonality strategy (Figure 2), demand for every end item is fulfilled by a single common product.

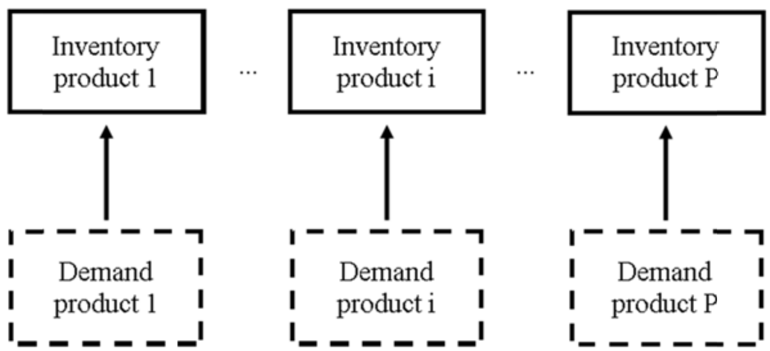


Fig. 1: No product commonality strategy

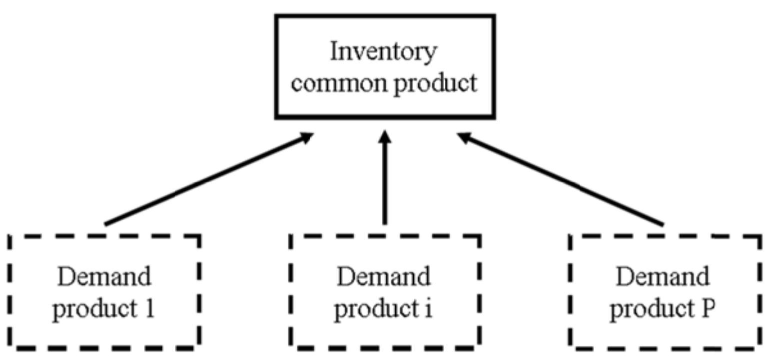


Fig. 2: Pure product commonality strategy

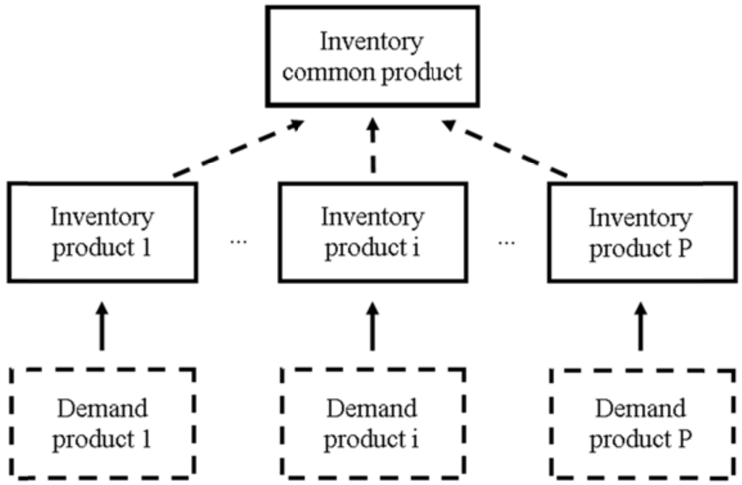


Fig. 3: Commonality as backup

With the *commonality as backup* strategy (Hillier 2002), the common product acts as a backup for all the unique products: demand will initially be satisfied by the unique product, and the common product will only be used to fulfill the demand if the unique product stocks out (Figure 3).

A setting with component commonality implicitly assumes that assembly needs to take place: each product needs to be assembled from two or more components. Figure 4 illustrates the *no component commonality strategy*: every end item needs to be assembled from its own unique components (the figure assumes two components per end item). A *pure component commonality strategy* refers to a situation in which one unique component of every product is replaced by a single (typically more expensive) common component, while the other unique components continue to exist (Figure 5: in this example, all the b-type components are replaced by the common component).

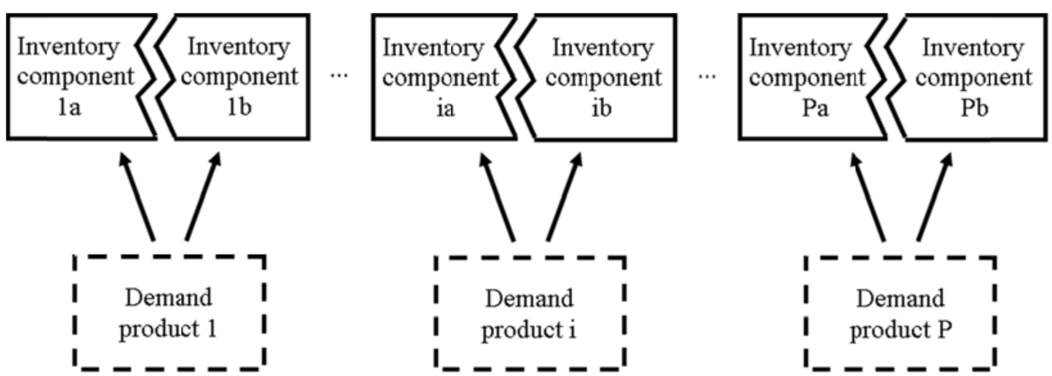


Fig. 4: No component commonality strategy

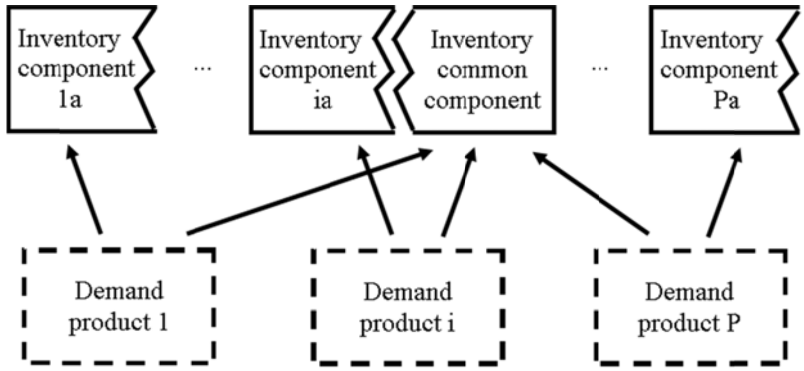


Fig. 5: Pure component commonality strategy

Hillier 1999 investigates the effect of product commonality in a multiperiod setting, considering multiple product types with stochastic demand. Two strategies are compared: the no product commonality strategy (Figure 3), i.e., the demand for every product is fulfilled by the unique product, and the pure product commonality strategy (Figure 4), i.e., the demand for every product is fulfilled by a common product. The objective is to find an optimal order up to level for every product type in view of minimizing the expected discounted cost, i.e., the sum of expected inventory holding costs, shortage costs for backlogged demand, and purchasing costs.

The multiperiod no commonality case is essentially similar to the single product multiperiod case, introduced in section II.A.2.a. Hillier 1999 uses a myopic policy to solve the multiperiod no commonality case. As discussed in section II.A.2.a, the result of this policy is optimal given that cost parameters and demand distributions do not change. As such, we can derive a closed-form expression for the critical fractile: for the multiperiod no commonality case, this boils down to expression (5).

The pure product commonality strategy is much harder to solve. As explained previously, two types of decisions have to be made: (i) before demand is observed, the optimal order policy of the common product needs to be determined (the *order decision*); and (ii) after demand is observed we need to decide on the amount of the common product that will be allocated to every demand (the *allocation decision*). As the allocation decision is only relevant when total demand exceeds the inventory of the common product, the allocation decision will only affect shortage costs. Hence, it is optimal to prioritize the allocation of the inventory based on the shortage cost (i.e., first allocate inventory to the product with the highest shortage cost), until all inventory of the common product is allocated.

Although it is straightforward to optimally allocate the common product to the demands, and although it can be proven that the myopic policy is optimal (Gerchak and Hening 1989), it is not easy to find a closed-form expression to determine the critical fractile. Hillier 1999 shows that, in case the shortage cost is the same for every end product, the critical fractile can be determined by expression (5), with the demand for the common product given by the sum of all individual product demands. For the general case (when shortage costs are different for every end product), the author proposes an approximation.

In addition, Hillier 1999 calculates the break-even cost for the single period and multiperiod case with homogeneous costs and demand distributions. This break-even cost reflects the (fictitious) purchase cost of the common product that would cause both strategies (i.e., the no commonality strategy and the pure commonality strategy) to be equally attractive (i.e., yielding the same minimal total cost). The results indicate

that, in the single period case, replacing unique products by a 10 to 30% more expensive common product can be cost-effective. However, in a multiperiod case, the break-even cost of the common product is only a few percentages more expensive than the unique product. Hence, in a multiperiod setting, the pure product commonality strategy will only be beneficial if the cost premium for the common component is small. The reason for this observed difference is the fact that, in the single period case, the use of pure commonality reduces the number of products to be purchased. In the multiperiod case (with backorders), the total number of products purchased remains the same as for the no commonality strategy. Hence, the pure commonality strategy only reduces holding cost in the multiperiod case, whereas it reduces both purchasing and holding cost in the single period case.
Hillier 1999 also illustrates that the break-even cost increases when either (i) the number of unique products being replaced increases, (ii) the shortage cost increases, (iii) the holding cost increases, or (iv) the variability of demand increases. Furthermore, (as might be expected) the benefit of component commonality decreases with increasing demand correlation.

The research of Hillier 1999 was further extended in Hillier 2002 by introducing a third strategy: the commonality as backup strategy (Figure 3). The aim remains to determine the optimal order up to level for every product in view of minimizing the sum of expected purchasing costs, holding costs and shortage costs, in a multiperiod setting. While Hillier 1999 assumed that unmet demand is backlogged, Hillier 2002 assumes that unmet demand is lost.
Again, two decisions need to be made: the order up to levels need to be determined before demand is observed, and the common item needs to be allocated once demand is known. The interaction between the stocking levels of the unique items and common item makes it difficult to solve: indeed, the demand distribution faced by the common item depends on the order up to level of the unique items.
For the commonality as backup strategy, it turns out to be impossible to find a closed-form expression for the optimal stocking level. Only for the special case of two products with homogeneous costs and uniform demand distributions, it is possible to determine the expected cost per period (though not the critical fractile). Hillier 2002 shows that the commonality as backup strategy is a better strategy than the no commonality or pure commonality strategy, in the sense that it allows to lower the expected total costs. The reason is that the no commonality and pure commonality strategy are essentially special cases of the commonality as backup strategy.
The commonality as backup strategy is reduced to the no commonality strategy when the order up to level of the common product is set to zero. When the order up to levels of all unique products are set to zero, the commonality as backup strategy is reduced to the pure commonality strategy. Hence, the optimal cost of the commonality as backup strategy will always be at least as favorable as the optimal cost of the no commonality or pure commonality strategy. Additionally, Hillier 2002 performed a number of numerical experiments, indicating that the break-even cost for the commonality as backup strategy is always higher than for the pure commonality strategy.

Hillier (2000) investigated the effect of component commonality. Two strategies are compared: the no component commonality strategy (Figure 4), and the pure component commonality strategy (Figure 5). The results and insights derived for the component commonality case (Hillier 2000) are similar to the ones obtained for the

product commonality case (Hillier 1999, 2002):
(a) In the absence of commonality, it is always optimal to use equal order-up-to levels for the unique components of the same product. The critical fractile is again represented by expression (5); the holding cost consists of the sum of the holding costs of all the unique components of the given product, and the purchasing cost consists of the sum of all purchasing costs.
(b) For the pure component commonality setting, a greedy algorithm similar to Hillier 1999 solves the allocation decision. No closed-form equation for the critical fractile is found. However, a lower and upper bound for the critical fractile can be derived.
(c) Numerical experiments indicate that the break-even cost of the common component turns out to be much lower in the multiperiod case than in the single period case (the argument has been explained when we described product commonality).
(d) Numerical experiments indicate that component commonality is more beneficial when (i) there are more products in which the component is used, (ii) the shortage costs increase, (iii) the coefficient of variation of demand increases, (iv) demands are more negatively correlated, (v) holding costs increase, or (vi) delivery lead times increase.

Hale et al. 2001 present a model for a single period, two product setting with one-way substitution of components. Both end products (one high quality product and one low quality product) are assembled from two unique components (see Figure 6). The unique components of the high quality product are more expensive than the unique components of the low quality product. One component of the high quality product can act as a substitute for one unique component of the low quality product.

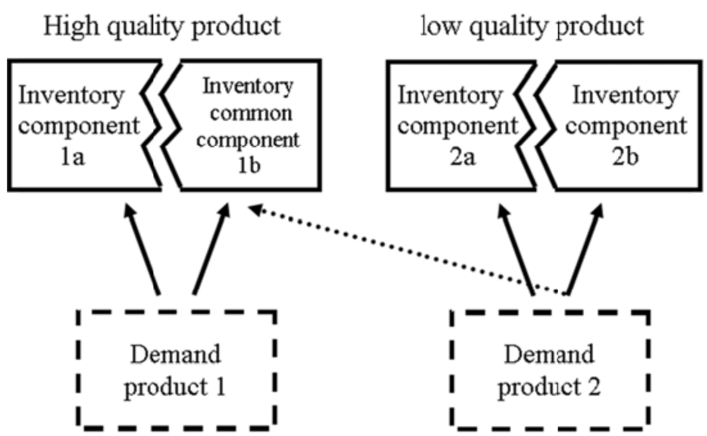


Fig. 6: Component substitution

The objective is to maximize the total profit: i.e., the sum of sales revenue plus salvage value of unsold inventory minus purchasing cost of the components. The problem again boils down to a 2-stage decision process. At the start of the period (before demand is observed), the order quantity of every component has to be decided. After demand is known, an allocation decision has to be taken, i.e., how much of the substitute component is used to manufacture the high quality end product and how much is used as a substitute to manufacture the low quality end product. It is assumed that the revenue and cost parameters are such that it is optimal to only use the substitute component to fulfill demand of the low quality product if the low

quality component stocks out. Consequently, the allocation decision can be solved by a greedy algorithm. Hale et al. 2001 proved that it is optimal set the order up to level of the substitute component (1b) equal to the order up to level of the unique component (1a). This is a striking result: at first sight, it seems obvious to use a higher order-up-to level for the substitute component, as it can also be used in the low quality end item. However, any additional units of substitute component (1b) ordered on top of the order up to level of the unique component (1a) can only be substituted, and can never used to assemble the high quality product. Given that component (2b) is less expensive, this can never be optimal. Note that, surprisingly, Hale et al. 2001 do not consider inventory holding cost; consequently, they do not take into account any risk pooling benefits.
Hale et al. 2001 prove that the profit function is concave in the order-up-to levels, such that the globally optimal order-up-to levels are found by solving the first order conditions. From these first order conditions a closed-form expression can be derived that resembles the critical fractile of the basic newsvendor model, and which balances the total expected overage cost with the total expected underage cost. This expression can be used to determine lower and upper bounds for the optimal order-up-to levels; the order-up-to levels can, however, not be derived analytically.

### *b. Flexible resources*

Van Mieghem 1998 and Van Mieghem and Rudi 2002 consider flexibility in terms of flexible resources. A company has to decide on the amount of capacity to invest in product-dedicated resources and flexible resources, before demand for the different products is known (the *investment decision*). After demand is observed, the company has to decide which product volumes will be produced on the product-dedicated resources, and which will be produced on flexible resource (the *allocation decision*).
There is a large resemblance between the resource flexibility setting and the commonality as backup strategy (Hillier 2002). While the commonality as backup strategy contains dedicated products and a backup product, the setting with flexible resources uses dedicated resources and flexible resources.

Van Mieghem 1998 determines the optimal capacity levels of the dedicated resources and the flexible resource in view of maximizing total profit (i.e., operating profit minus capacity cost) for a single period case with only 2 products. Each product can be manufactured on the product-dedicated resource, or on the (more expensive) flexible resource. Note that, in contrast with the commonality settings discussed in Section II.A.3.a, the flexible resource setting does not consider holding costs or shortage costs.
Van Mieghem 1998 develops a two-stage decision process. The allocation decision is determined through a linear programming model, that maximizes operation profits under demand and capacity constraints, assuming that the capacity of the different resources is already specified and demand is already known.

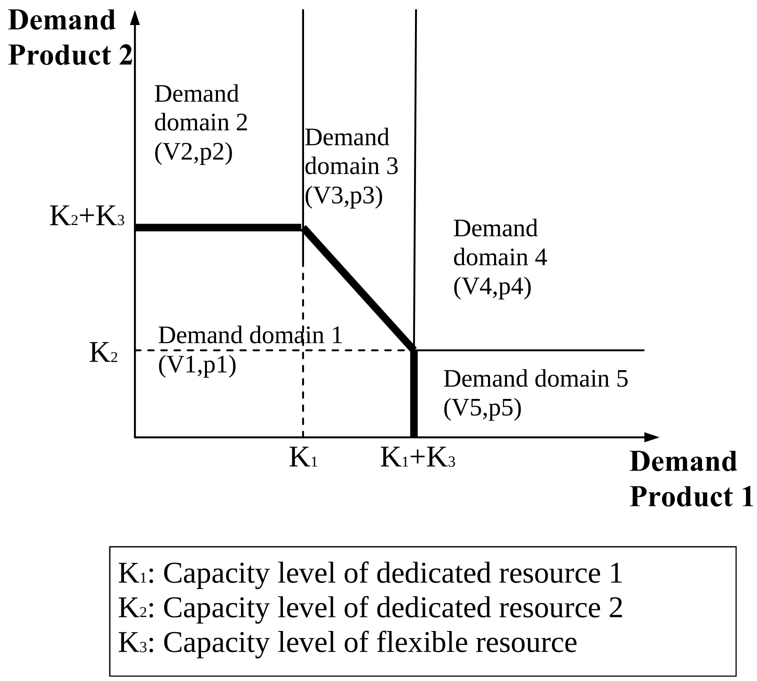


Fig. 7: Demand domains with constant shadow prices

In a second stage, the shadow prices of the capacity constraints (denote by a vector $\boldsymbol{V}$) are used to examine the impact of a change in the capacity levels on the operation profits. Since the demand space can be divided into 5 domains with constant shadow prices (see Figure 7), the expected shadow price for predetermined capacity levels is given by:

$$\sum_{i=2}^{5} \boldsymbol{V}_i p_i$$

with $p_i$ the probability that demand falls in domain $i$.
Using this expected shadow price, the unit capacity costs (vector $\boldsymbol{k}$) and the fact that the total profit function is concave, it is possible to derive the optimal capacity levels from the first-order conditions of the total profit function (equation 9).
More specifically, capacities should be set such that the following optimality condition holds:

$$\sum_{i=2}^{5} \boldsymbol{V}_i p_i = \boldsymbol{k} \tag{9}$$

Amazingly, there is a large resemblance with the original newsvendor problem. The left-hand side of equation (9) shows the total expected shadow price (expected underage cost), which equals the probability that demand cannot be fulfilled, multiplied with the cost of not meeting demand (i.e., lost operational profit). The right-hand side represents the vector of unit capacity costs, i.e., the cost of investing in an additional unit of capacity (expected overage cost). In the optimum, the expected profit increase should be equal to the expected cost increase. It turns out that the associated optimal capacity vector coincides with a critical fractile of a multivariate demand distribution (similar to the original newsvendor model).

The main insights derived are the following:
(a) It is either optimal to (i) only invest in dedicated resources, (ii) invest in the dedicated resource of product with the highest profit margin and in the flexible resource, or (iii) invest in all three resources. Other scenarios can never be optimal.
(b) The capacity of the flexible resource depends only on the level of uncertainty in demand. If we can reduce the uncertainty in demand, it is less necessary to invest in an expensive flexible resource.
(c) Even in the case of perfectly positive correlation between demands, it can be optimal to invest in the flexible resource. The reason is that the flexible resource gives the option to change the demand allocation, and to manufacture more units of the highest-margin product.

Van Mieghem and Rudi 2002 extend the single period multidimensional newsvendor model of Van Mieghem 1998 to a single period newsvendor network. The main difference is that, in this paper, both a resource decision (invest in product dedicated versus flexible resources) and an inventory decision (invest in dedicated versus flexible raw material) need to be taken.
The authors propose a two-stage decision process. In the first stage, both the order-up-to levels of the raw materials and the capacities of the resources are decided before demand is observed. In the second stage (the processing activity), after demand has been observed, a linear programming model determines which raw material and which resource is used to produce the end products.
The objective is to find the optimal order up to levels and capacity levels that maximize the total profit, which consists of the total sales margin minus the sum of total shortage cost, total holding costs, total purchasing costs and total investment costs. It is straightforward that adding flexibility (either in terms of a flexible resource or a flexible raw material) will never decrease the optimal profit.
The solution method is similar to the one proposed in Van Mieghem 1998. A linear programming model is developed to determine the optimal allocation of raw materials and resource capacities to end products, given the demand realizations, order up to levels and capacity levels. Shadow prices are used to determine the impact of increasing the capacity of a resource, or increasing the order up to level of a product, on the total operating profit (vector $\boldsymbol{V_k}$ and $\boldsymbol{V_S}$, respectively). The demand space can be divided into domains with constant shadow prices. Since total profit is jointly concave (in resource capacities as well as order up to levels), it is sufficient to calculate the first order conditions of the total profit function. Capacities and order up to levels should be placed such that the following optimality conditions hold:

$$\textstyle\sum_i \boldsymbol{V}_{ki} p_i = \boldsymbol{k} \tag{10}$$

$$\textstyle\sum_i \boldsymbol{V}_{Si} p_i = \boldsymbol{c} + \boldsymbol{h} \tag{11}$$

Similar to Van Mieghem 1998, a critical fractile of a multivariate demand distribution can be derived, with at the left-hand side the expected shadow price with respect to resource capacity (expression 10) or order up to level (expression 11). In other words, the left-hand side of expression 10 (resp. 11) represents the profit that is lost if capacity (resp. order up to level) is too small. This represents the expected underage cost. The right-hand side of expression 10 (resp. 11) represents the expected overage cost. This is the expected cost of investing in one additional unit of capacity (unit

capacity costs in expression 10) or purchasing one additional product (unit purchasing $\boldsymbol{c}$ and holding cost vector $\boldsymbol{h}$ in expression 11).
Van Mieghem and Rudi 2002 show that, if we assume that demand follows a bivariate normal distribution, the optimal total profit is increasing with average demand and decreasing with variance of demand.

### 4. Limitations of the newsvendor model

Despite its popularity in the literature, the newsvendor model suffers from a number of drawbacks/limitations. In this section, we'd like to draw attention to these limitations, and explain why the newsvendor model often falls short of modeling real-life settings.
A first limitation is that any order placed is received immediately. This essentially boils down to a zero lead time, which is a very unrealistic assumption in real life.
Secondly, the optimal demand fractile depends on the "unit shortage cost", which is very hard to estimate in real life. Finally, although closed-form expressions of the critical fractile can be found for complex problems, no analytical expression can be found to derive the optimal order up to levels for multiproduct newsvendor models with flexibility. Other methods, such as simulation, can then be utilized to determine the optimal order policy. Simulation is described in next section.

## *B. Simulation*

Simulation is commonly used to model and analyze stochastic systems. The basic principle of simulation is to imitate the behavior of a real life system by the use of computer simulation software (Kelton et al. 2007). Essentially, simulation models can be used to study the behavior of any type of complex system, irrespective of the assumptions. As such, this methodology is much more widely applicable than most analytical methodologies (such as, e.g., the Markov models discussed in Section III).

Given its versatility, simulation has been used in a number of research papers to study optimal order-up-to policies for inventories with flexibility. Table 3 gives an overview of the key references. As shown, simulation has been used to optimize the inventory control parameters in settings with transshipment, product substitution and postponement. These settings are in fact highly related: in each setting, flexible inventory is used as a backup in case the "original" product is out of stock.
In general, two approaches have been used to determine optimal order-up-to levels using simulation: the gradient based approach, or exhaustive search. For further technical details on these approaches, the reader is referred to the appendix of this paper. In section 1 below, the references on product substitution listed in Table 3 are covered in some further detail. The papers on transshipment and postponement are briefly touched upon in sections 2 and 3.

| Reference | Kind of flexibility | Time Horizon | Description | Objective |
|---|---|---|---|---|
| Robinson 1990 | Transshipment problem | Multi period | - Lost sales or backlogged demand<br>- Discounted cost<br>- Gradient based approach | Minimize sum of holding, shortage and transshipment cost |
| Herer et al. 2006 | Transshipment problem | Multi period | - Backlogged demand<br>- Average cost<br>- Gradient based approach | Minimize sum of purchasing, holding, shortage and transshipment cost |
| Khouja et al. 1996 | Product substitution | Single period | - Two product types<br>- Two-way substitution<br>- Exhaustive search approach | Maximize sum of sales revenue and salvage value minus purchasing cost |
| Bassok et al. 1999 | Product substitution | Single period | - Multi product<br>- One-way substitution<br>- Full downward substitution<br>- Gradient based approach | Maximize sum of sales and salvage value minus purchasing, holding, shortage and substitution costs. |
| Tibben-Lembke and Bassok 2005 | Postponement | Single & multi period | - Multiple regular products & 1 generic product<br>- Discounted cost<br>- Gradient based approach | Maximize sum of sales and salvage value minus purchasing, holding, shortage and customization cost |

Table 3: Overview of key references that study inventory flexibility by means of simulation

## 1. Product substitution

A product substitution setting consists of a set of multiple products, with the possibility to use a product from the set as a substitute whenever the original product runs out of stock. The product substitution literature can be divided in two categories: *customer driven* substitution and *decision-maker driven* substitution. In the first case, it is commonly assumed that the customer has an exogenously determined probability of purchasing a substitute if the preferred product is out of stock. The company only needs to determine the optimal order policies for the different products; no allocation decisions have to be made (e.g. Mahajan and Van Ryzin. 2001, McGillivary and Silver 1978, Parlar and Goyal 1984, Pasternack and Drezner 1991 and Netessine and Rudi 2003). By contrast, in a decision-maker driven substitution model, the company needs to determine both the optimal order policies and the optimal allocation of the substitute to demand (e.g Khouja et al. 1996, Bassok et al. 1999). The papers on substitution listed in Table 3 only cover decision-maker driven substitution.
Khouja et al. 1996 have investigated the ability of substitution for a two product, single period case. Both products can act as a substitute of each other when the inventory is not sufficient to fulfill the demand of the period. The objective of this research is to find the optimal order up to level that maximizes the expected profit,

which consists of the sales revenue plus the salvage value of inventory at the end of a period minus the purchasing cost. Since the problem is too complex to solve analytically, Khouja et al. 1996 provide a simulation approach to identify the optimal order quantity. They can limit the search space by calculating a lower and upper bound on the optimal order up to levels. As no proof of concavity can be provided, the optimal order up to levels are found by exhaustively examining the limited search space.

Bassok et al. 1999 extend the substitution model to a single period, multiproduct model with full downward substitution, i.e., unmet demand at the end of a period can be satisfied by on hand inventory of *any* product with higher quality . The aim is to find the optimal order policy for all the products in view of maximizing total profit (i.e., sales revenue plus salvage value of inventory at the end of the period minus purchasing, holding, shortage and substitution costs). The profit function is concave and differentiable in the order up to levels. The problem can be formulated as a two-stage decision problem. Before demand is observed, orders have to be placed for every end product. After demand is observed, the allocation of inventory to demand is done. The allocation decision is solved by a greedy algorithm: the basic idea is to allocate as much as possible to the product with the highest revenue minus substitution cost. The order decision is solved approximately by an iterative algorithm, using the order-up-to levels from the newsvendor model as initial values, and iteratively adjusting these levels until the first order condition of the profit function is satisfied. Numerical experiments were conducted to uncover the relationship between different parameter settings and the resulting benefit of substitution. As expected, the benefit of substitution increases with higher demand variability and lower substitution cost.

## 2. Transshipment problems

Transshipment problems have been widely studied in the literature (an overview can be found in Paterson et al. 2009 and Lang 2010). In a transshipment problem, we consider a company with multiple locations. Every location has a stochastic demand, which it tries to satisfy by its on-hand inventory. If a location does not have enough inventory to satisfy demand, an emergency (lateral) transshipment can take place from another location with positive on-hand inventory (Figure 8).

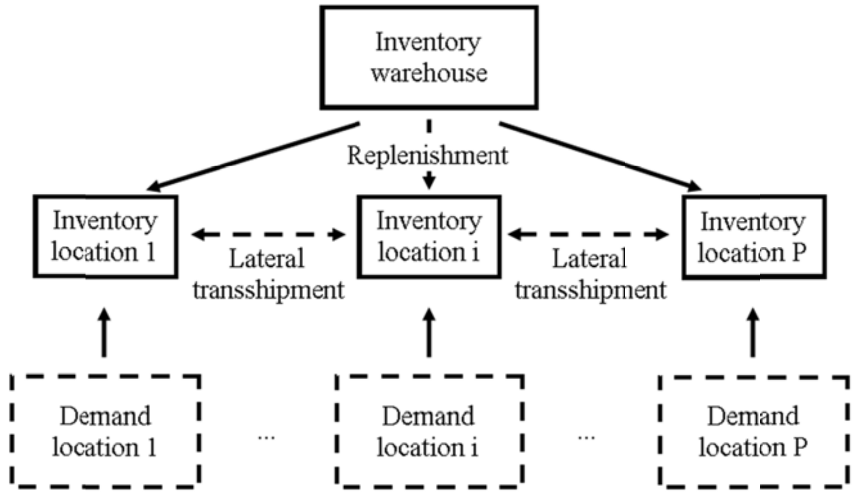


Fig. 8: Transshipment problem

In the majority of the literature, the transshipment time is assumed to be negligible. The periodic review transshipment problem is highly similar to the multiproduct newsvendor problem. In particular, there is a large resemblance with the component substitution problem of Hale et al. 2001. In the transshipment problem, optimal order up to levels have to be chosen, at every location, before demand is observed. Later, when demand is observed, the optimal emergency transshipments have to be determined. In Hale et al. 2001, the same decisions have to be taken albeit at component level instead of location level. The main difference is that the transshipment problem explicitly considers flexibility costs (i.e., transshipment cost) while these are not considered in the component substitution problem. Also, in the transshipment problem, there is no need to assemble any components.

## 3. Postponement

In the postponement setting of Tibben-Lembke and Bassok 2005, multiple regular (or "dedicated") products exist along with one generic (or "flexible") product. Demand can be satisfied by the regular product or by the (more expensive) generic product, which can be customized to the specific demand in zero time. Both a single period case and a multiperiod case are considered, with the objective of maximizing profit ( total revenue plus total salvage value minus purchasing cost, holding cost, shortage cost and customization cost). Again, this is a two-stage decision problem: before demand is observed, the order quantities have to be decided. Once the demand of the period has been observed, the inventory of the generic product needs to be allocated. Tibben-Lembke and Bassok 2005 suggested a greedy algorithm, which allocates the generic product in decreasing order of the products' net profit, i.e., the sum of revenue and shortage cost minus customization cost (note that this net profit is the additional profit captured when unmet demand is satisfied by the generic product). Because the expected profit function is concave, a base stock policy is optimal, independent of the initial inventory. To find the optimal order-up-to levels, a gradient based approach is used. The algorithm starts from the newsvendor solution for every regular product and zero inventory for the generic product. The first order condition is calculated for every product (regular and generic). If the first order condition of the generic product is strictly positive, the order up to level of generic product and the regular products are updated iteratively until the first order conditions of all the products are zero.

# III. CONTINUOUS REVIEW INVENTORY SYSTEMS

In a continuous review inventory system, the inventory position of the items is checked continuously, and a replenishment order is triggered when this inventory position falls at or below a given reorder level (Chopra and Meindl 2007). In the literature, both (S-1,S) and (R,Q) policies have been studied. We focus our overview on relevant papers which assume *complete pooling* (i.e., no part of the inventory is reserved to only satisfy dedicated demand)[1].

| Paper | Kind of flexibility | Description | Objective |
|---|---|---|---|
| Bayindir et al. 2005 | Product substitution | - Exponential replenishment lead time<br>- 2 products<br>- One-way substitution<br>- Lost sales<br>- (S-1,S) model | Maximize sales revenue minus the sum of purchasing and inventory holding cost |
| Liu and Lee 2007 | Product substitution | - Exponential replenishment lead time<br>- 2 products<br>- One-way substitution<br>- Backlog<br>- Substitution upon demand arrival and order delivery<br>- (S-1,S) model | Investigate effect on average inventory, average backorder, fill rate and average substitution probability of demand |
| Olsson 2010 | Transshipment | -Exponential replenishment lead time<br>- Unidirectional lateral transshipment with n locations<br>- Both backlogs and lost sales are allowed<br>- (S-1,S) and (R,Q) policy | Minimize sum of holding costs, backorder (or lost sales) cost and transshipment cost |

Table 4: Overview of selected papers dealing with flexibility in continuous review inventory systems

It should be noted that all papers assume that the demands for the different items are uncorrelated, and unit-sized (i.e., each customer order consists of 1 single unit). The arrival process of customer orders is assumed to be Poisson. This makes the problem setting amenable to analysis through continuous-time Markov chains. In general, the

[1] Systems that do reserve part of the substitute inventory for dedicated demand typically use a "threshold level": no substitution is allowed when the substitute inventory drops at or below this threshold. This is called *partial pooling*. Examples of such papers (which are not covered here) include Hadley and Whitin 1963, Köchel 1996, Axsäter 2003.

state of the system represents the net inventory (i.e., on hand inventory minus backorders) of a product type or location. Transition rates are represented by demand arrival rates and order arrival rates. The main drivers of the objective function (such as, e.g. average inventory, average unmet demand, average order quantity, average holding cost, average purchasing cost and average shortage cost) can be evaluated using the steady state probabilities. As such, the Markov chain methodology can be used as a "calculation engine" in a broader algorithm that aims to optimize the parameters of the inventory control policy.

Bayindir et al. 2005 consider a one-way substitution (S-1,S) system with only two products. The objective of their research is to find the optimal order up to levels that maximize total profit, i.e., sales revenue minus the sum of purchasing cost and inventory holding cost. A two-dimensional Markov process is developed: the first dimension represents the on hand inventory of product 1 and the second dimension represents the on hand inventory of product 2[2]. An example of the state space is depicted in Figure 9 ($\lambda_i$ refers to the demand rate of product i, $\mu_i$ refers to the replenishment rate of product i, and $S_i$ refers to the order-up-to level of product i). In this example, product 2 serves as the substitute.

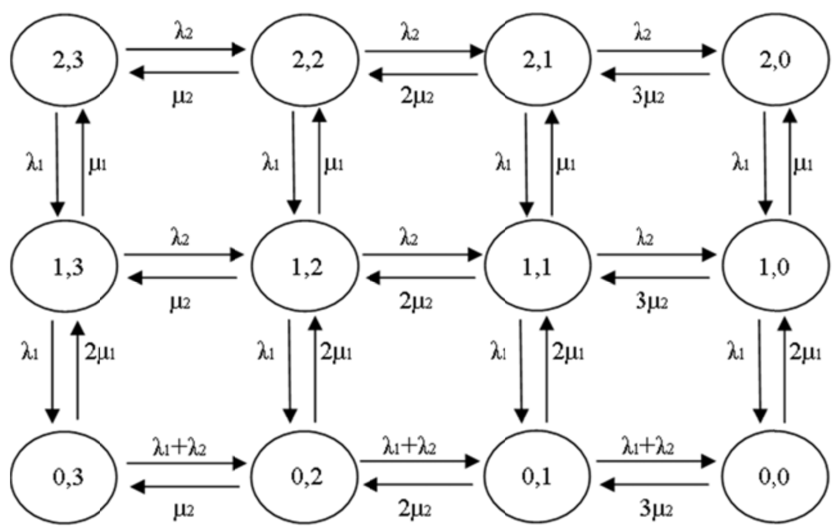


Fig. 9: State space with $S_1$=2, $S_2$=3 and product 2 as the substitute

As long as there is a strictly positive on hand inventory for both products, the demand rates of product 1 and product 2 are equal to $\lambda_1$ and $\lambda_2$, respectively. When product 1 has no on hand inventory and product 2 still has strictly positive on hand inventory, the demand rate for product 2 is equal to $\lambda_1 + \lambda_2$ (because product 2 is used as a substitute). Note that the rate at which a replenishment of a particular product is received depends on the amount of units ordered of that product.
Numerical experiments are conducted to analyze the expected profit function (determined as expected revenue minus expected purchasing and holding costs), and

[2] Note that taking flexibility into account results in increasing the number of dimensions. Without flexibility, the inventory system could have been modeled as two one-dimensional Markov chains.

to find the optimal order up to levels. As the expected profit function is not concave, the optimal order up to level are found through an exhaustive search. One interesting result is that substitution does not always improve optimal expected profit. When the demand rate of the substitute is larger than the replenishment rate of the substitute, allowing substitution will worsen the expected profit. The reason is that the replenishment rate is even not sufficient to satisfy the “own” demand with a high profit margin; allowing substitution results in using this scarce inventory to satisfy the demand of a product with a lower profit margin. The optimal expected profit increases when the replenishment rate of the substitute increases.

Liu and Lee 2007 developed a multiproduct (S-1,S) inventory system with backlogs. They propose three different policies to use one-way substitution. In a first policy, one-way substitution is only used when demand arrives for a product and there is no inventory left of that product. Note that this policy assumes that, once the demand is backlogged, it continues to be backlogged until the next order of that product is received (even when the inventory of the substitute becomes strictly positive in the meantime). Though this assumption is in fact not realistic (in practice, it is logical to also allow one-way substitution upon order delivery), the authors state that it is the usual assumption in continuous-review models with backlogged demand (see also Olsson 2010). In contrast, the second policy allows one-way substitution upon demand arrival *and* upon order delivery of the substitute. The third policy is highly similar to the second policy: the only difference is that one-way substitution upon order delivery is only allowed if the backlogged demand is lower than a target level. Though the methodology is developed for only two products, it can be extended to a multiproduct case (although the resulting multi-dimensional Markov model will be more difficult to solve).

Olsson 2010 developed approximate models for a continuous review inventory system in which one-way lateral transshipments are allowed. In the backorder case, both a (S-1, S) policy and a (R, Q) policy are considered. In the lost sales case, only the (S-1, S) policy is analyzed. The aim of the research is to find the optimal order parameters which minimize the expected cost (i.e. the sum of holding cost, shortage cost and lateral transshipment cost). Since lateral transshipments are allowed, the demand rate at a location in fact depends on the net inventory of all locations. Olsson (2010) approximates the demand rate of a location such that it only depends on the net inventory of that location, which simplifies the calculations necessary to obtain the steady-state probabilities and total expected cost. The optimal order parameters are determined through exhaustive search.

# IV. CONCLUSION

This paper has discussed multiple methodologies used to determine optimal order policies for inventory systems with substitution.
Three distinct methods have been covered, which are closely linked to the type of inventory system used. In a periodic review inventory system, newsvendor models and Monte Carlo simulation are the most popular methodologies. Though the newsvendor model is attractive due to its elegance and mathematical tractability, it turns out to be rather stringent in its assumptions (in particular, the assumption of zero replenishment lead times is hard to defend in real-life situations). Moreover, research

has shown that the actual computation of the order up to levels is not straightforward for systems with substitution. Consequently, for complex settings, researchers tend to switch to simulation models to analyze the optimal policies.
By contrast, continuous review inventory systems are commonly analyzed by means of continuous-time Markov chains. Unfortunately, all papers using this methodology focus on uncorrelated demands, and restrict attention to exponentially distributed lead times, limiting the generalization of the results (indeed, other correlation structures and/or different lead time distributions may have a large influence on the optimal order policies). Moreover, it is usually assumed that substitution can only occur at the moment of demand arrival, while it is in fact more realistic to also allow substitution upon order arrival.
Consequently, it appears that the literature on inventory systems with substitution (though ample in nature) shows a gap: i.e., to the best of our knowledge, no model seems to have succeeded in taking into account nonzero correlations and strictly positive (possibly stochastic) lead times simultaneously. Combining both aspects is a highly relevant issue to adequately represent real-life settings. In future research, we aim to bridge this gap.

## APPENDIX: TECHNICAL DETAILS ON OPTIMIZATION VIA SIMULATION

Assume we want to minimize a general function *g(y)*, defined as the expected value of a random function *G(y,w)* which depends on the random variable *w*:

$$min_{y \epsilon Y}\{g(y) = E[G(y,w)]\} \qquad \text{(A.1)}$$

We can consider this as a two-stage stochastic programming problem, with *w* the stochastic variable and *y* the decision variable. Before *w* is known, we need to determine *y* such that *g(y)* is minimized. Once *w* has been observed and decision *y* has been taken, the optimal decision minimizing *G(y,w)* can be determined. Optimizing the function *G(y,w)* is referred to as the second stage optimization problem. Minimizing *g(y)* in terms of *y* is then referred to as the first stage optimization problem.

When the possible realizations of *w* are finite $(w_1,\ldots,w_M)$, with corresponding probabilities of occurrence $p_k$ $(k=1,\ldots,M)$, *g(y)* can be readily determined:

$$g(y) = \sum_{k=1}^{M} p_k\,[G(y,w_k)]$$

However, when the number of possible realizations is large (possibly infinite), the optimization problem can be more efficiently solved through Monte Carlo simulation. The general idea is to draw a random sample $w^1, \ldots, w^N$ and use this sample to estimate *g(y)* by the corresponding sample average $\hat{g}(y)$:

$$\hat{g}(y) = \sum_{i=1}^{N}\left[G(y,w^i)\right]/N$$

$\hat{g}(y)$ is an unbiased estimator of *g(y)*, i.e. $\hat{g}(y)$ will converge to *g(y)* if N goes to infinity.

The original problem (expression A.1) can then be approximated by

$min_{y \epsilon Y}\{\hat{g}(y)\}$

and is referred to as the *sample average approximation* (Shapiro et al. 2009) problem.

The multiproduct newsvendor problem with flexibility can be seen as a two-stage stochastic programming problem. In the first stage problem, the optimal order quantities need to be determined before demand is observed. After demand is realized, the optimal allocation of the flexible product needs to be decided (i.e., the second stage problem).

The *exhaustive search* approach finds $min_{y \epsilon Y}\{\hat{g}(y)\}$ by exhaustively examining all possible $y \in Y$. This approach is used by Khouja et al. (1996), but can only be used efficiently when the number of possible options for $x$ is small.

When the number of options is large, the *gradient based approach* (Shapiro 2000) is more efficient. For $g(y)$ and $G(y,w)$ convex and differentiable in $y$, we can estimate the derivative of $g(y)$:

$$\nabla_{\mathrm{y}} g(y) = \nabla_{\mathrm{y}} E[G(y,w)] = \sum_{i=1}^{N} \left[\nabla_{\mathrm{y}} G(y,w^i)\right]/N = \nabla_{\mathrm{y}} \hat{g}(y)$$

with $\nabla_{\mathrm{y}} \hat{g}(y)$ an unbiased estimator of $\nabla_{\mathrm{y}} g(y)$. In other words, we can draw a sample of realizations of the random variable $w^i$. For every $w^i$ we solve the deterministic second stage optimization problem. Because $G(y,w)$ is differentiable in $y$, $\nabla_{\mathrm{y}} G(y,w^i)$ can be calculated. Averaging $\nabla_{\mathrm{y}} G(y,w^i)$ over all realizations gives us an estimator of the gradient of $g(y)$. Since $g(y)$ is convex, we can update y until $\nabla_{\mathrm{y}} \hat{g}(y) = 0$, yielding an estimation of the optimal $y^*$.

The gradient based approach is more efficient than the exhaustive search approach, but can only be used if $g(y)$ and $G(y,w)$ are convex and differentiable in $y$. A variety of papers, described in the next section, use this approach (e.g. Robinson 1990, Herer et al. 2006, Bassok et al. 1999 and Tibben-Lembke and Bassok 2005).

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